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Integrability of generalized Ermakov systems

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Abstract. We find all (four) first integrals for two-dimensional Ermakov systems.

Athorne (1991) showed that the structure of generalized Ermakov systems introduced by Ray and Reid (1979), namely

$$\ddot{x} + \omega^2(t)x = \frac{1}{x^3} f(y/x) \tag{1a}$$

$$\ddot{y} + \omega^2(t)y = \frac{1}{y^3} g(y/x) \tag{1b}$$

is essentially that of an autonomous Hamiltonian system. He was presumably motivated by the fact that the original Ermakov system (Ermakov 1880), comprising the time-dependent harmonic oscillator

$$\ddot{x} + \omega^2(t)x = 0 \tag{2a}$$

and the Ermakov–Pinney equation (Ermakov 1880, Pinney 1950)

$$\ddot{y} + \omega^2(t)y = 1/y^3 \tag{2b}$$

is a Hamiltonian. However, in general, Ermakov systems are not Hamiltonian. We look at definitely non-Hamiltonian Ermakov systems of the form

$$\ddot{r} - r\dot{\theta}^2 = \frac{1}{r^3} f(\theta, r^2\dot{\theta}) \tag{3a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r^3} g(\theta, r^2\dot{\theta}) \tag{3b}$$

(the presence of the $r^2\dot{\theta}$ makes the system non-Hamiltonian) and show the existence of an invariant of Ermakov-type (Ermakov 1880). (The change to plane polars is suggested by the fact that the Ermakov invariant is of angular-momentum type. We carefully distinguish between Ermakov invariants and Lewis invariants (Lewis 1967, 1968) despite a tendency in the literature to conflate them. Whereas an Ermakov invariant has the nature of a generalization of angular momentum, a Lewis invariant has the nature of an energy integral.)

Generalized Ermakov systems are characterized by their Lie algebra, $s\ell(2, R)$, (Leach 1991). A well known example of an Ermakov system in three dimensions is the classical magnetic monopole (Govinder *et al* 1993). This example exhibits invariance under, in

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addition to $s\ell(2, R)$, the generators of the Lie algebra $so(3)$. In general, Ermakov systems in three dimensions are non-Hamiltonian. For our purposes we concentrate on two dimensions as the increase in mathematical complexity for three dimensions is not compensated by further insight into the problem.

We utilize the group-theoretic approach for finding first integrals. This requires the knowledge of the symmetries of (3) which were given by Leach (1991) as

$$G_1 = \frac{\partial}{\partial t} \tag{4a}$$

$$G_2 = 2t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \tag{4b}$$

$$G_3 = t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r}\,. \tag{4c}$$

The first extension of G_1, namely

$$G_1^{[1]} = \frac{\partial}{\partial t} \tag{5}$$

gives the functional form of its first integrals as

$$I = I(u, v_1, w_1, x) \tag{6}$$

where

$$u = \theta \qquad v_1 = r \qquad w_1 = r\dot{r} \qquad x = r^2\dot{\theta}\,.$$

(We use these characteristics rather than $r, \theta, \dot{r}$ and $\dot{\theta}$ to maintain as close a resemblance as possible with the characteristics of G_2 and G_3.) The requirement that

$$\dot{I} = 0$$

results in a partial differential equation with associated Lagrange's system

$$\frac{\mathrm{d}u}{x} = \frac{\mathrm{d}v_1}{v_1 w_1} = \frac{\mathrm{d}w_1}{x^2 + f(u, x) + w_1^2} = \frac{\mathrm{d}x}{g(u, x)}\,. \tag{7}$$

Taking the first and fourth terms gives us a first integral of Ermakov-type

$$\tilde{I} = M(u, x)\,. \tag{8}$$

(See also Govinder and Leach 1993.) The first and third terms of (7) give the Riccati equation

$$\frac{\mathrm{d}w_1}{\mathrm{d}u} - \frac{w_1^2}{N} - N - \frac{f(u, N)}{N} = 0 \tag{9}$$

where N is obtained by inverting (8) to give

$$x = N(I, u)\,. \tag{10}$$

We can, in principle, integrate (9) to obtain another first integral

$$J_1 = P_1(w_1, u, I) \tag{11}$$

say. The first two terms of (7) can be easily integrated to give

$$K_1 = \frac{1}{v_1^2}\exp\left(2\int\frac{Q_1}{N}\mathrm{d}u\right) \tag{12}$$

where Q_1 is obtained by inverting (11) to give

$$w_1 = Q_1(J_1, u, I)\,. \tag{13}$$

Using the first extension of G_2 we obtain the functional form of its first integrals as

$$\tilde{J} = \tilde{J}(u, v_2, w_1, x) \tag{14}$$

where

$$u = \theta \qquad v_2 = rt^{-1/2} \qquad w_1 = r\dot{r} \qquad x = r^2\dot{\theta}\,.$$

Proceeding in a similar manner to before we again obtain $\tilde{I}$ and J_1 (in the latter case the Riccati equation obtained is identical to (9)). The only new first integral is

$$K_2 = \frac{1}{v_2^2}\exp\left(2\int\frac{Q_1}{N}\mathrm{d}u\right) - \int\left[\frac{1}{N}\exp\left(2\int\frac{Q_1}{N}\mathrm{d}u\right)\right]\mathrm{d}u\,. \tag{15}$$

$G_3^{[1]}$ requires its first integrals to have the functional form

$$\tilde{K} = \tilde{K}(u, v_3, w_2, x) \tag{16}$$

where

$$u = \theta \qquad v_3 = rt^{-1} \qquad w_2 = r\dot{r} - r^2t^{-1} \qquad x = r^2\dot{\theta}\,.$$

As expected we again obtain $\tilde{I}$. The other two first integrals are

$$J_2 = P_2(w_2, u, I) \tag{17}$$

and

$$K_3 = \frac{1}{v_3^2}\exp\left(2\int\frac{Q_2}{N}\mathrm{d}u\right)\,. \tag{18}$$

Here J_2 is the solution of the Riccati equation

$$\frac{\mathrm{d}w_2}{\mathrm{d}u} - \frac{w_2^2}{N} - N - \frac{f(u,N)}{N} = 0\,. \tag{19}$$

We have obtained six first integrals for the system (3). As we know that only four independent first integrals exist, we have to relate two to the other four. We know that

$$v_1^2 = v_3^2/t^2\,. \tag{20}$$

We can thus solve (12) in terms of v_1 and substitute for v_1, $Q_1 = w_1$ and $Q_2 = w_2$ in (18) to obtain

$$K_3 = \frac{1}{K_1}\,. \tag{21}$$

We also know that

$$\begin{aligned} w_2 &= r\dot{r} - r^2t^{-1} \\ &= w_1 - v_2^2\,. \end{aligned} \tag{22}$$

Substituting (22) into (19) results in the Bernoulli equation (Ince 1956)

$$\frac{\mathrm{d}(v_2^2)}{\mathrm{d}u} - \frac{2w_1v_2^2}{N} + \frac{v_2^4}{N} = 0 \tag{23}$$

which, upon substituting

$$v_2^2 = 1/y \qquad u = x$$

becomes the linear first-order equation

$$y' + \frac{2w_1}{N}y - \frac{1}{N} = 0\,. \tag{24}$$

We can solve (24) to obtain

$$y = \left(A + \int \frac{1}{N} \exp\left(2\int \frac{Q_1}{N}\mathrm{d}u\right)\mathrm{d}u\right)\exp\left(-2\int \frac{Q_1}{N}\mathrm{d}u\right) \tag{25}$$

and, inverting and substituting for y, w_1 and w_2, we obtain

$$A = \frac{1}{v_2^2}\exp\left(2\int \frac{Q_1}{N}\mathrm{d}u\right) - \int \left[\frac{1}{N}\exp\left(2\int \frac{Q_1}{N}\mathrm{d}u\right)\right]\mathrm{d}u \tag{26}$$

which is just K_2. Thus (19) gives us no new information about the system (3).

We therefore have that (3) has the four independent first integrals

$$\tilde{I} = M(u, x) \qquad (27a)$$

$$J_1 = P_1(w_1, u, I) \qquad (27b)$$

$$K_1 = \frac{1}{v_1^2}\exp\left(2\int \frac{Q_1}{N}\mathrm{d}u\right) \qquad (27c)$$

$$K_2 = \frac{1}{v_2^2}\exp\left(2\int \frac{Q_1}{N}\mathrm{d}u\right) - \int \left[\frac{1}{N}\exp\left(2\int \frac{Q_1}{N}\mathrm{d}u\right)\right]\mathrm{d}u\,. \qquad (27d)$$

We note that, if we had let the arbitrary functions f and g in (3) depend on θ only, then the system (3) would be Hamiltonian provided

$$f(\theta) = -2\int g(\theta)\mathrm{d}\theta \tag{28}$$

(Leach 1991). However, if we relax the condition (28), we still obtain the four first integrals (27) with $M(u, x)$ in (27a) explicitly given by

$$M = \tfrac{1}{2}(r^2\dot{\theta})^2 - \int g(\theta)\mathrm{d}\theta\,. \tag{29}$$

This is just the Ermakov invariant.

We finally remark that Cervero and Lejarreta (1991) have found invariants for Hamiltonian Ermakov systems. Their development is somewhat complicated. We have adopted an approach similar to that used above to find all (four) first integrals for Hamiltonian Ermakov systems (Govinder and Leach 1994). The parallels between the Js and Ks in this paper and the Ks and Js in that paper are rather intriguing and bear further investigation.

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